Statistical modeling of extreme events in space

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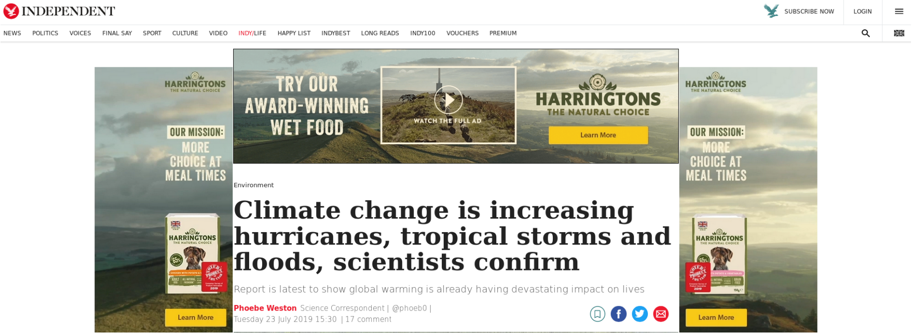
Climate change and extreme events



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Environment

Climate change is increasing hurricanes, tropical storms and floods, scientists confirm

Report is latest to show global warming is already having devastating impact on lives

Phoebe Weston Science Correspondent | @phoeb0 |
Tuesday 23 July 2019 15:30 | 17 comment

Extreme events are located in space (s) and time (t) ...

s =“Montpellier (FR)”, t=“30 September 2014”

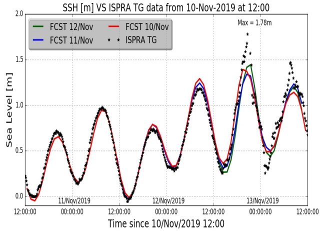
... and they could be dependent.

s =“Venice (IT)”, t=“8 November 2014”

Outline of the talk

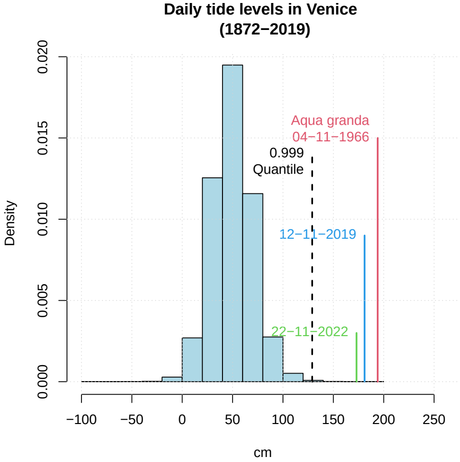
"My" data...

Venice (IT), 12 November 2019



Sea level	Area of Venice submerged (percent)
+90 cm	1.84%
+100 cm	5.17%
+110 cm	14.04%
+120 cm	28.75%
+130 cm	43.15%
+140 cm	54.39%
+150 cm	62.98%

Extrapolation



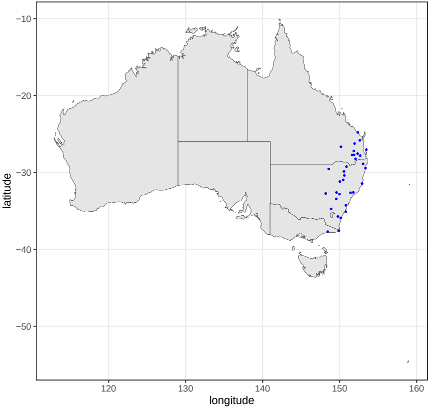
- Can a standard statistical model help ?
- Guess the **p-quantile** or **return level** with a return period of $T = 1/p$ (years) i.e. value that Y overcomes with "small" $0 < p < 1$

“Ordinary” vs "Extreme value" statistics

- **"Ordinary" statistics**: Tries to describe main part of distribution; may ignore outliers.
- **"Extreme value" statistics**. Tries to characterize the tail of the distribution; keeps only the extreme observations.
- Fits **asymptotically-justified** distributions to characterize the tail.
- Uses only a subset of the data considered to be extreme.
  - GEV - models **block maxima**
  - GPD - models **threshold exceedances**
- Dependence described differently not via covariances or correlations, but looking at the tail.
- Spatial process are not Gaussian...

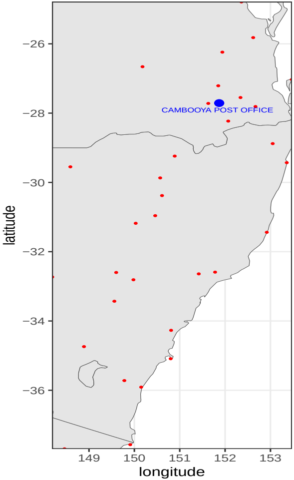
Example: Australian rainfall data

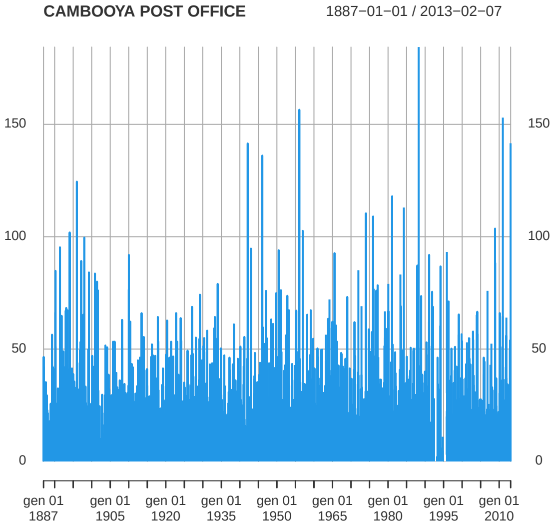
- Daily rainfall data from 33 stations in the East of Australia (1841-2013) (web source)



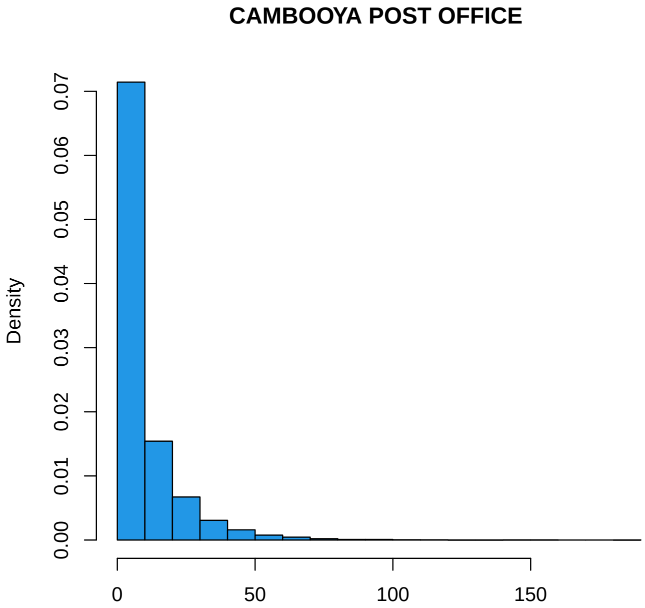
Lavery, B., Joung, G. and Nicholls, N. (1997). *Australian Meteorological Magazine*, **46**, 27-38.

Australian rainfall data: one station





Australian rainfall data: one station distribution

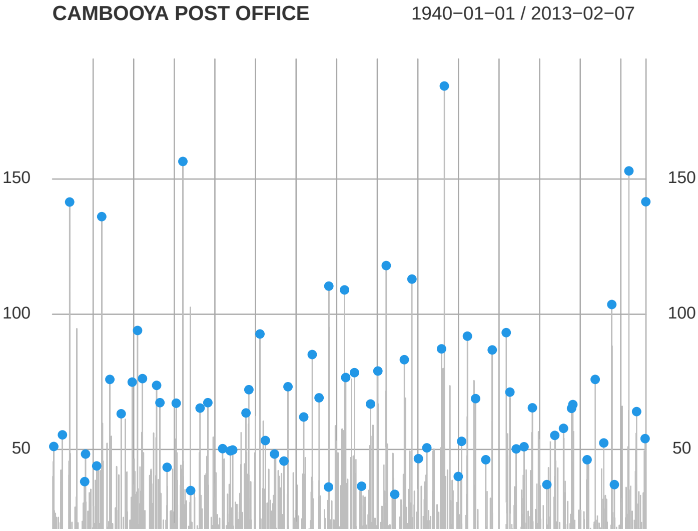


Classical extreme value models

Coles, S. (2001) *An Introduction to Statistical Modeling of Extreme Values* , Springer

Block maxima

Focus on the annual maxima (block maxima)



Principles of stability

Fundamental to all characterizations of extreme value processes is the concept of **stability**.

- For example, we might propose one model for the annual maximum of a process, and another for the 5-year maximum. Since the 5-year maximum will be the maximum of 5 annual maxima, the models should be mutually consistent.
- Similarly, a model for exceedances over a high threshold should remain valid (in a precise sense) for exceedances of higher threshold.
- The expression of such stability requirements as mathematical statements leads to asymptotic models.

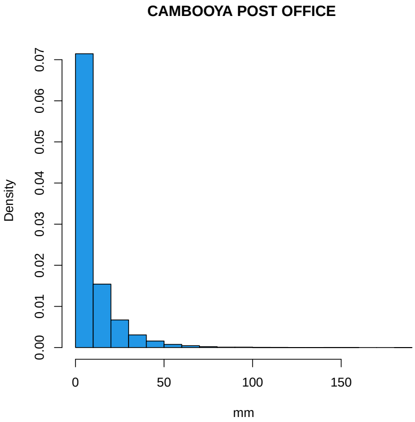
The Gaussian world...

- Gaussianity is a common assumption in statistical modeling
- Asymptotic justification: Central Limit Theorem
- **sums** $S_n = Y_1 + \ldots + Y_n$,

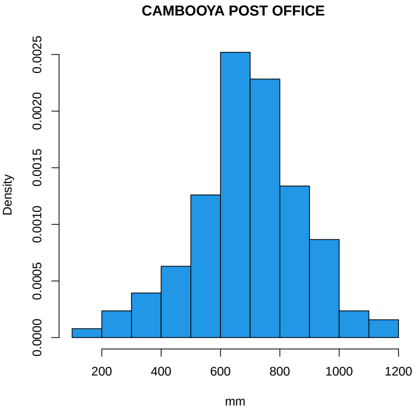
$$\frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{S_n - b_n}{a_n} \approx Z \sim \mathcal{N}(0, 1)$$

  for large n

Annual rainfall (sums)

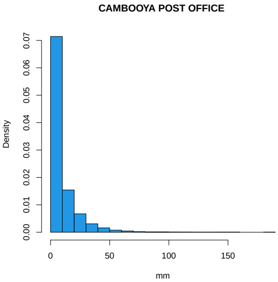


Y

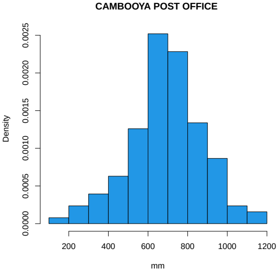


Total amount of rain over one year.

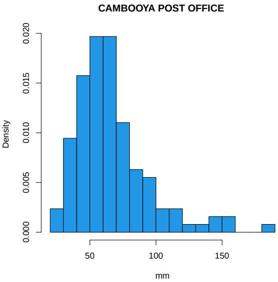
Annual rainfall (maxima)



Y



Total amount of rain over one year



Maxima of daily rain over one year

- A symmetric distribution cannot adequately model the maxima

Model for block maxima

- $Y_1, Y_2, \ldots$ sequence of independent random variables with common distribution F
- **Maxima** $M_n = \max(Y_1, \ldots, Y_n)$
- If there exist sequences of constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that,

$$\frac{M_n - b_n}{a_n} \approx Z \sim G$$

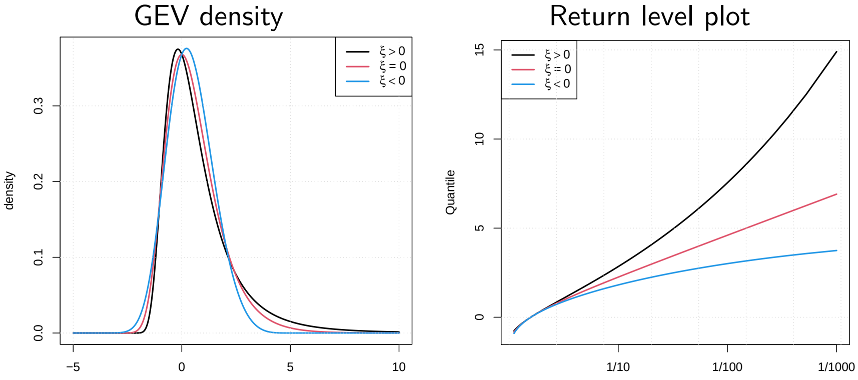
  for large n then the distribution G is necessarily of the form

Generalized Extreme Value (GEV) distribution

$$G(z) = \begin{cases} \exp\left[-\{1 + \xi(z-\mu)/\sigma\}_+^{-1/\xi}\right], & \xi \neq 0 \\ \exp\left[-\exp\{-(z-\mu)/\sigma\}\right], & \xi = 0 \end{cases}$$

- $\mu \in \mathbb{R}$ (location), $\sigma > 0$ (scale) and $\xi \in \mathbb{R}$ (**shape**), where $a_+ = \max(0, a)$
- support $\{z \in \mathbb{R} : 1 + \xi(z-\mu)/\sigma > 0\}$

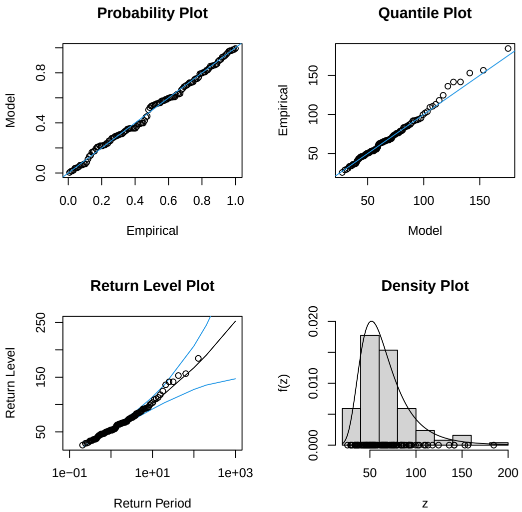
GEV: different shape parameters



$\mu = 0, \, \sigma = 1$

Annual rainfall (maxima): fitted model

$\widehat{\mu} = 53.84\,(1.86), \quad \widehat{\sigma} = 18.48\,(1.41), \quad \widehat{\xi} = 0.12\,(0.07), \quad n = 127$

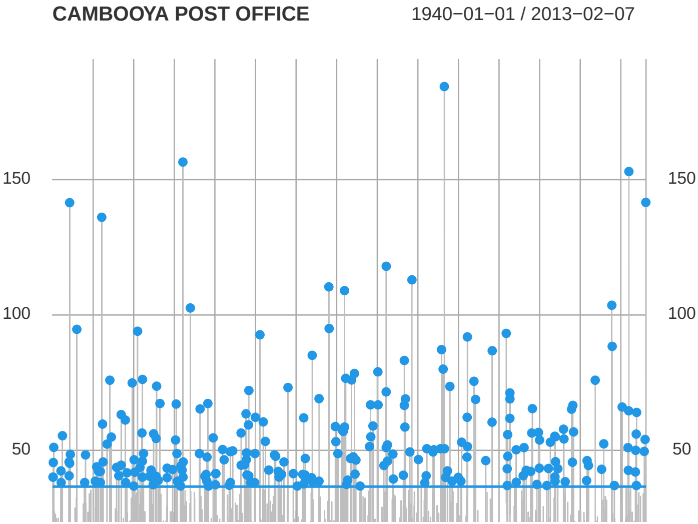


Block maxima: pros/cons

- Convention is to work with blocks equal in length to the calendar year, but what if you have hourly data collected over just a few years?
- Block length too small -limiting arguments will not hold (the GEV is a limiting result, which holds approximately for large n)
- Block length is too large - not enough maxima to work with!
- Possible sensitivity of GEV parameter estimates to block length
- Extremely wasteful of data:
  - Discard all but the block maxima
  - Often results in throwing away tens of thousands of observations some of which might be 'extreme' just not as extreme as the block maxima!

Peak Over Threshold (POT) analysis

Focus on the exceedances over a large threshold.



Model for threshold exceedances

Under the same condition for GEV, the conditional distribution of high threshold **exceedances**

$$Y - u | Y > u$$

may be approximated

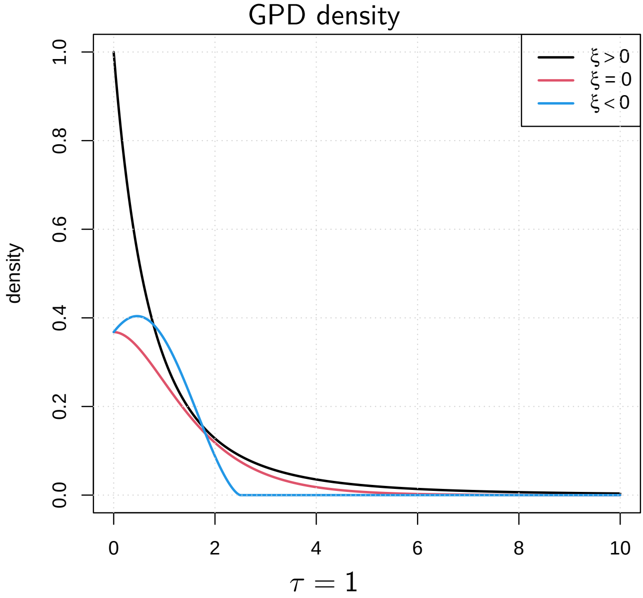
$$Y - u | Y > u \approx Z \sim H$$

for large u. The distribution H is necessarily of the form

Generalized Pareto (GP) distribution

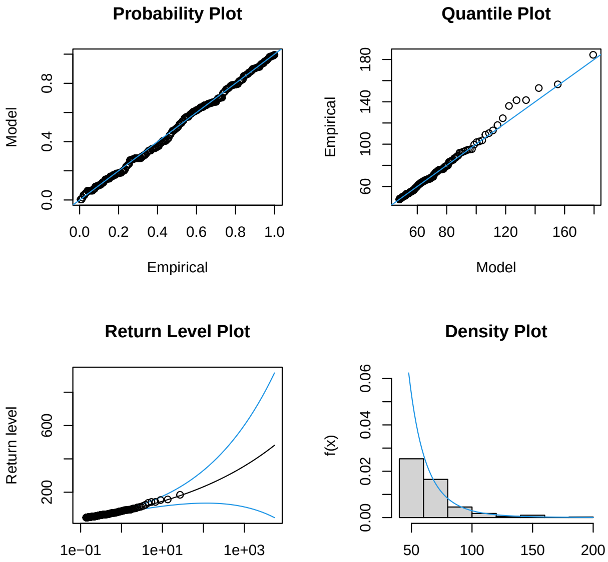
$$H(z) = \begin{cases} 1 - (1 + \xi z/\tau)_+^{-1/\xi}, & \xi \neq 0 \\ 1 - \exp(-y/\tau), & \xi = 0 \end{cases}$$

GPD: different shape parameters



Exceedances: fitted model

$\widehat{\tau} = 13.825\,(0.6087)$, $\widehat{\xi} = 0.11\,(0.03)$, $u = q_{0.975}$, $n = 1105$



Why GPD model ?

$=$ The shape parameters ξ in GEV and GP are equal

$=$ The scale parameters σ and τ are related

$$\tau \approx \sigma + \xi(u - \mu)$$

$+$ Estimation (inference) with more data

$+$/- Which threshold u ?

$+$/- In a cluster of exceedances data are temporally dependent.

$+$ Model for a real event

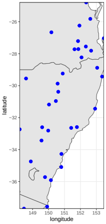
Which spatial analysis for extremes ?

Possible goals of extremal spatial analysis

- Pointwise maps of quantiles (return levels) - minimal information about extremes
- Long-run prediction of events, for insurance/planning, e.g. floods - need to take account of spatial dependence
- Short-range forecasting, e.g. avalanches, forest fires - need to include real-time information
- Simulation of events for input to other models, e.g. hydrological models for runoff in mountain valleys after extreme storms

Spatial setting

- In the spatial setting, assume that $\{Y_1(s)\}, \{Y_2(s)\}, \dots$ denotes a sequence of independent random processes defined over the region $S \in \mathbb{R}^d$
- Observations at a finite collection of sites $s_1, \dots, s_D$.



- Extreme events are sparse by definition
- We look at a single model that links the data at the different sites together.
- Spatial scale matters...

Spatially varying marginals (Spatial trends)

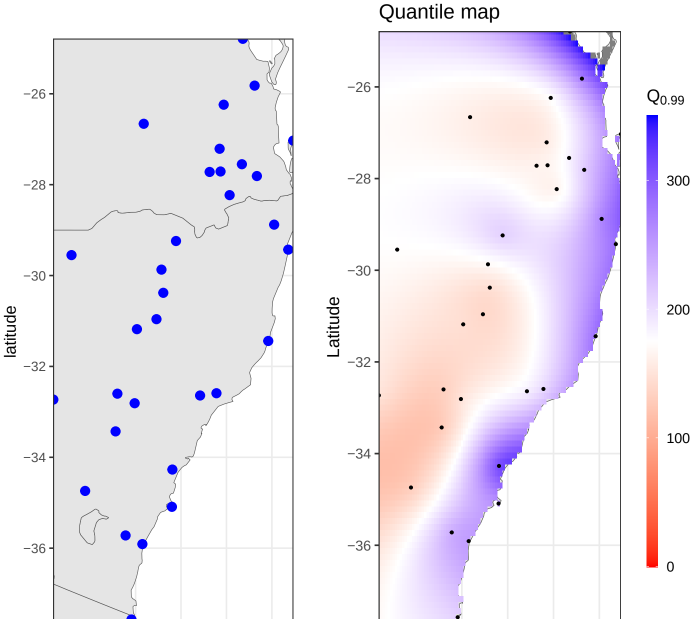
- The parameters of the marginal models depends on spatially-varying **covariates** (altitude, etc.).
- In practice, the tail index $\xi(s)$ can usually be assumed to be constant (or to vary smoothly) over an entire spatial region

Example:

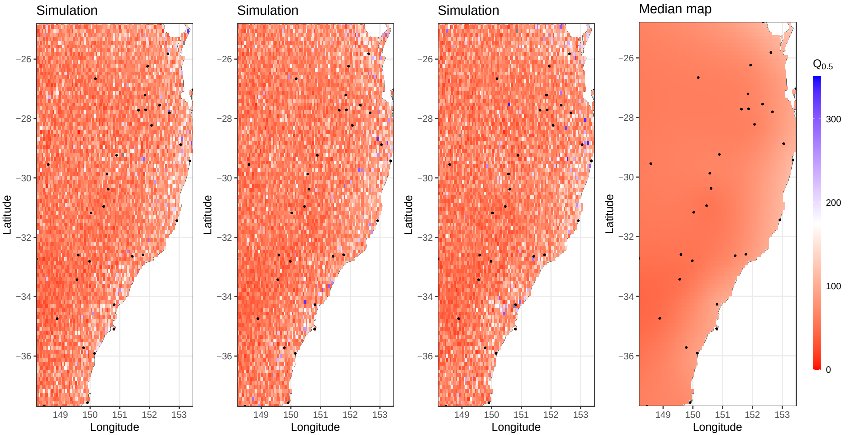
$$M_n(s) \sim GEV(y; \mu(s), \sigma(s), \xi(s))$$

$$\begin{aligned} \mu(s) &= h_\mu(\texttt{long}(s), \texttt{lat}(s)) \\ \log(\sigma(s)) &= h_\sigma(\texttt{long}(s), \texttt{lat}(s)) \\ \xi(s) &= \gamma_0 \end{aligned}$$

Australian rainfall data: spatial model fitting (1940-2013)



Australian rainfall: simulations of maxima



- Model definition implies independence among the sites !

Hierarchical modeling I

- see Opitz's talk
- Bayesian framework

Example :

Stage I: $Y(s)|\theta(s) \sim GEV(y; \mu(s), \sigma(s), \xi), \qquad \theta(s) = (\mu(s), \log \sigma(s))$

Stage II: $\mu(s) \sim \mathcal{GP}(\boldsymbol{u}(s)'\alpha, K_\mu(s, s'; \psi_1)),$

$$\log \sigma(s) \sim \mathcal{GP}(\boldsymbol{v}(s)'\beta, K_\sigma(s, s'; \psi_2))$$

Stage III: Prior on α, β, σ, ξ, and $\psi = (\psi_1, \psi_2)$

- Conditional on latent process $\theta(s)$, observations $Y(s)$, for $s \in S$ follow an extremal distribution

Hierarchical modeling II

Properties:

+ computationally feasible for large-scale problems using standard simulation techniques (Metropolis-Hastings algorithm, Gibbs sampling, . . .)
+ possibility of estimating quantiles spatially
- all extremal dependencies are incorporated through $\theta(s)$
- marginal distributions are not extremal
- episodic modelling/simulation difficult

Gaussian anamorphosis

- Remove spatial and temporal trend by fitting GEV (on maxima) or GP (on exceedances)
- Use this fit to transform data to "Gaussian" data (probability integral transformation)
- Apply standard geostatistics (see Ribatet's course)
- Back-transformation to original data

Properties:

\+ Easy using (R) software (`SpatialExtremes`, `evgam`)

\+ Gaussianity not essential (could be uniform, or `t`)

\- Distribution of joint extremes may be badly modelled because of properties of Gaussian model

+/- equivalent to use of (extremal) copula

Measuring the spatial dependence of extremes

- When dealing with spatial extremes, the correlation is no longer useful since the variance (and even the mean) might not even exist.

extremal coefficient function, $\theta(h)$

.

$$\Pr\left(M(s) \leq z, M(s+h) \leq z\right) = \Pr(M(s) \leq z)^{\theta(h)},$$

- The latter is a measure of the spatial dependence

$$1 \leq \theta(h) \leq 2,$$

since

- $\theta(h) = 1$: complete dependence
- $\theta(h) = 2$: independence

F-madogram

- In (conventional) geostatistics the semi-variogram is widely used

$$\gamma(h) = \frac{1}{2}\mathbb{E}(Y(s+h) - Y(s))^2$$

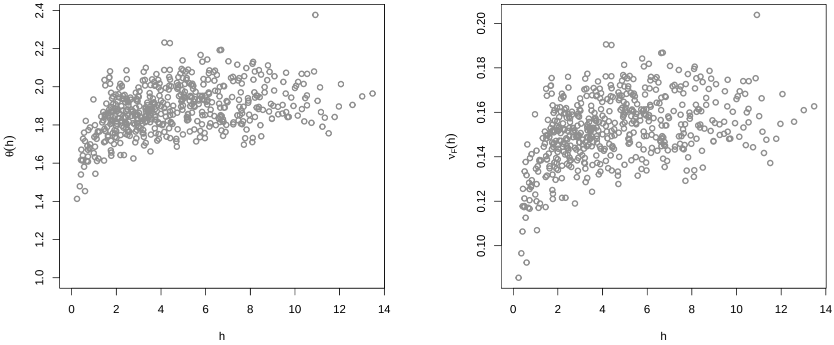
- The F-madogram (Cooley et al., 2006)

$$\nu_F(h) = \frac{1}{2}\mathbb{E}|F\{M(s+h)\} - F\{(M(s)\}|$$

satisfies

$$0 \leq \nu_F(h) = \frac{\theta(h) - 1}{\theta(h) + 1} \leq 1/3.$$

Australian rainfall: measures of the spatial dependence of extremes



Asymptotic models for spatial extremes

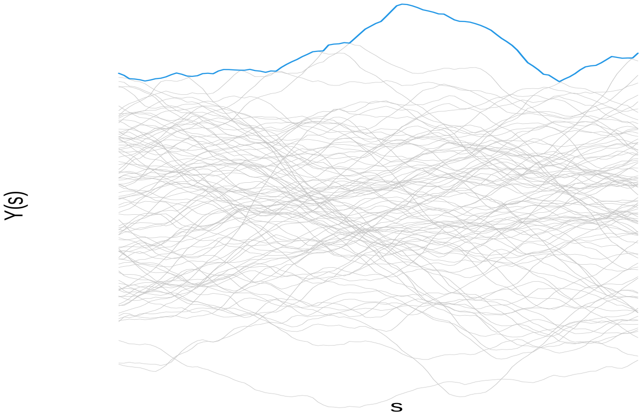
A. C. Davison, A.C., Padoan, S.A. and M. Ribatet, M. (2012) Statistical modeling of spatial extremes, *Statistical Science*, **27**, 161-186

Ferreira, A., and de Haan, L. (2014). The generalized Pareto process; with a view towards application and simulation. *Bernoulli*, **20**, 1717–1737.

Max-stable process I

- Max-stable process: infinite-dimensional generalization of GEV definition

$$\frac{\max_{i=1,\dots,n} Y_i(s) - b_n(s)}{a_n(s)} = \frac{M_n(s) - b_n(s)}{a_n(s)}, \quad s \in S$$



Max-stable process II

Let $\{Y(s)\}_{s \in S}$ be a stochastic process with continuous sample paths, $Y_1, Y_2, \ldots$ independent copies of it and define $M_n(s) = \max_{i=1,\ldots,n} Y_i(s)$. If

$$\left\{ \frac{M_n(s) - b_n(s)}{a_n(s)} \right\}_{s \in S} \xrightarrow{D} \{M(s)\}_{s \in S}, \qquad n \to \infty,$$

and $\{M(s)\}$ is assumed to be non degenerate, then $\{M(s)\}$ is **max-stable process**

Max-stable process: definition

A stochastic process $M(s)$, $s \in S$ is max-stable if there exist real functions $a_n(\cdot) > 0$ and $b_n(\cdot)$ such that

$$\frac{\max_{1 \le i \le n} Z_i(s) - b_n(s)}{a_n(s)} = M(s), \qquad s \in S$$

where $Z_1(\cdot), \ldots, Z_n(\cdot)$ are independent copies of $M(\cdot)$.

Model definition

- Pre-transforming marginal distributions simplifies the theory
- Common choice is unit **Fréchet distribution**
- i.e. GEV $\mu = 0$, $\sigma = 1$, $\xi = 1$

$$\Pr\{Z \leq z\} = \exp(-1/z), \qquad z > 0$$

- Models are mainly defined on this scale therefore we need to transform the data $M_n(s) = \max_{i=1,\dots,n} Y_i(s)$

$$M_n(s) \longrightarrow \tilde{M}_n(s) = -\frac{1}{\log F_s(M_n(s))}$$

- ⚠ ... we pretend to know the marginal the marginal GEV F_s.

Spectral representation (de Haan, 1984)

Let $\{M(s)\}_{s\in\mathbb{R}^d}$ be a max-stable process (unit Fréchet margins)

$$M(s) = \sup_{i\geq 1} R_i W_i(s), \quad s \in S$$

where $R_1, R_2, \ldots$ points on a Poisson point process on $[0, \infty]$ with intensity $r^{-2}dr$, and $W_1(s), W_2(s), \ldots$ be independent copies of a non-negative stochastic process with mean one,

- Suggests simulation algorithms for $M(s)$
- A rainfall-storms interpretation ?
  - R_i : storm strength ;
  - $W_i(s)$ spatial variation of the storm strength ...

Max-stable process: common model

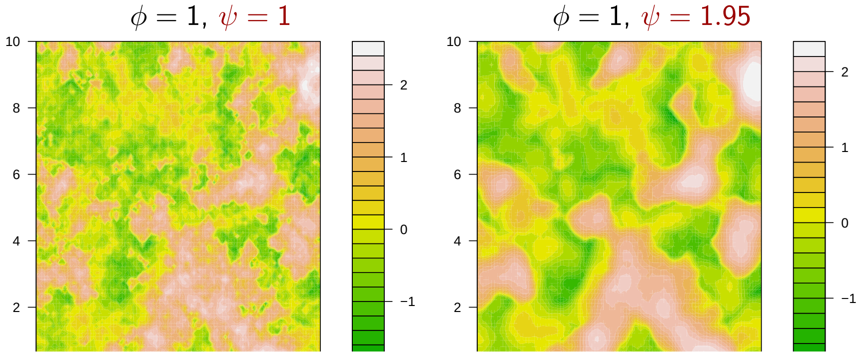
Specifying the W process in different ways, various max-stable models can be constructed:

- Schlather model (Schlather, 2002)
- Brown–Resnick model (Kabluchko et al., 2009)
- Extremal-t model (Opitz, 2013)

Schlather model

$$M(s) = \text{const} \times \sup_{i \geq 1} R_i \left[\max\{0, G_i(s)\}\right], \qquad s \in S, s \in S,$$

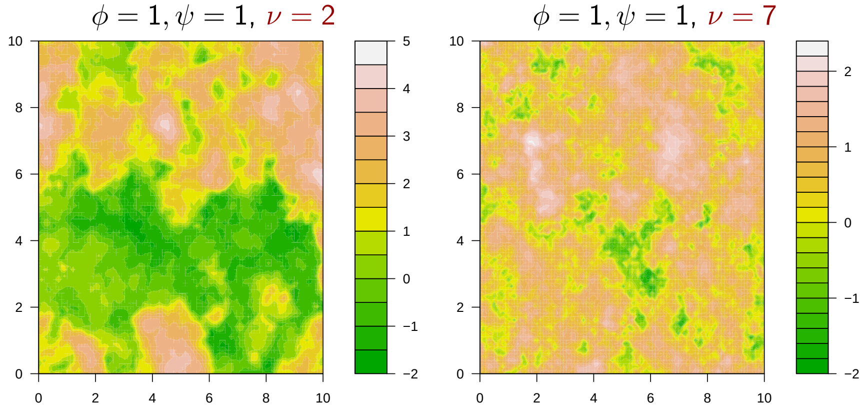
- G_i are independent replicates of a standard Gaussian process with correlation function ρ.
- $\rho(s, s') = \exp\left\{-\left(\|s - s'\|/\phi\right)^{\psi}\right\}$, $\phi > 0, 0 < \psi \leq 2$.
- Useful package in R: `mev`, `SpatialExtremes`, `RandomFields`



Extremal-t model

$$M(s) = \text{const} \times \sup_{i \geq 1} R_i \left[\max\{0, G_i(s)\}\right]^{\nu}, \qquad s \in S$$

- $\rho(s, s') = \exp\left\{-\left(\|s - s'\|/\phi\right)^{\psi}\right\}$, $\phi > 0, 0 < \psi \leq 2$.



$\theta(h)$ for max-stable process I

Recall the definition

$$\Pr\left(M(s) \leq z, M(s+h) \leq z\right) = \Pr(M(s) \leq z)^{\theta(h)},$$

- **Schlather**

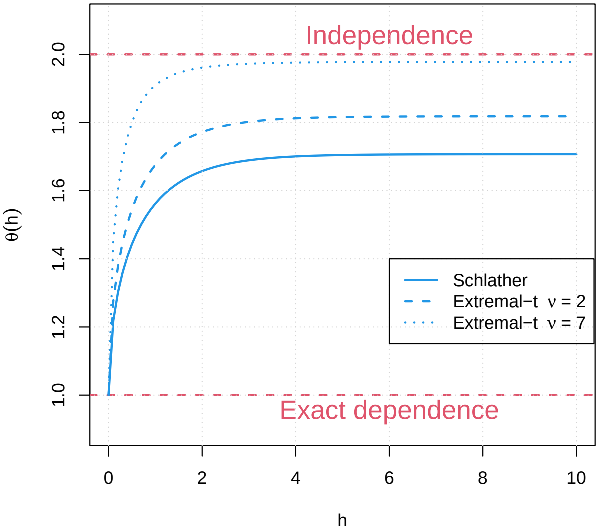
$$\theta(h) = 1 + \sqrt{\frac{1-\rho(h)}{2}}$$

- **t-extremal**

$$\theta(h) = 2T_\nu\left\{\sqrt{\frac{(1-\rho(h))(\nu+1)}{1+\rho(h)}}\right\}$$

T_ν CDF of a Student random variable with ν d.o.f.

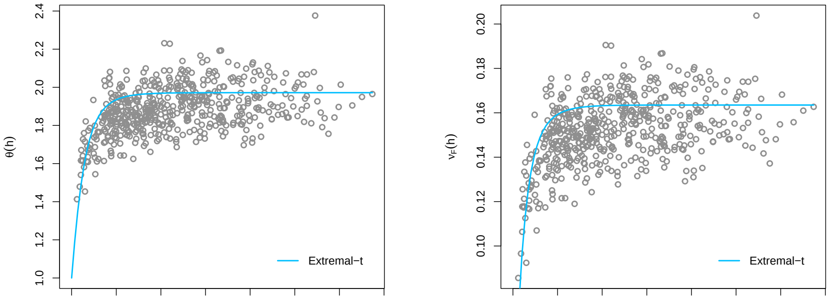
$\theta(h)$ for max-stable process II



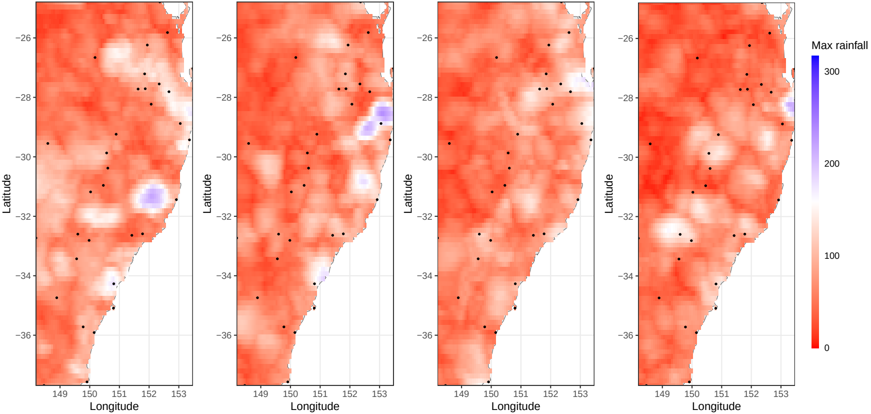
Australian rainfall: model fitting

$$
\begin{aligned}
M_n(s) &\sim GEV(y; \mu(s), \sigma(s), \xi(s)) \\
\mu(s) &= h_\mu(\texttt{long}(s), \texttt{lat}(s)) \\
\log(\sigma(s)) &= h_\sigma(\texttt{long}(s), \texttt{lat}(s)) \\
\xi(s) &= \gamma_0
\end{aligned}
$$

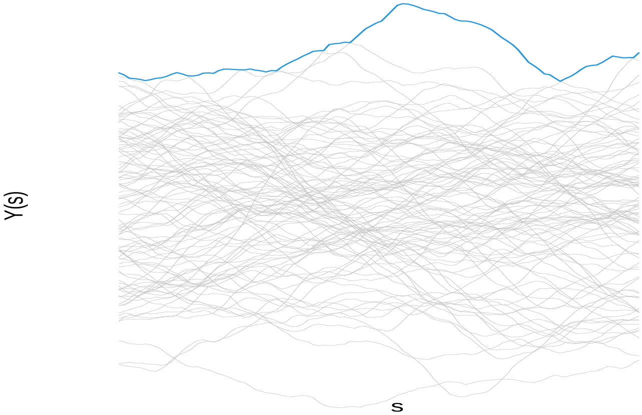
+ Extremal-t



Australian rainfall: simulations



Are max-stable processes realistic models ?

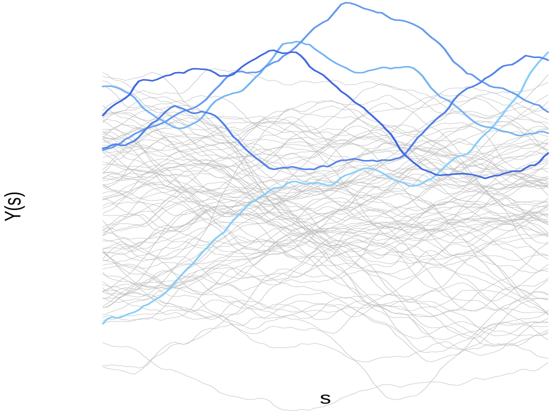


- Composition of a number of different events !

Looking at specific trajectories ...

- Conditioning event $\{\sup_{s \in S} Y(s) > u\}$, for large u
- We look at the law of

$$\{Y(s),\, s \in S \mid \sup_{s \in S} Y(s) > u\}$$



Model definition

- Theory simpler when marginal distributions are standard **Pareto distribution** ($M \sim \text{Pareto}(1)$)

$$\Pr\{M \leq z\} = 1 - 1/z, \qquad z \geq 1$$

- Models are mainly defined on this scale therefore we need to transform the data

$$Y(s) \longrightarrow \tilde{Y}(s) = \frac{1}{1 - F_s(Y(s))}$$

- ⚠ … we pretend to know the marginal distributions $F_s(Y(s))$

Generalized Pareto (GP) process: formal definition

- Spatial extension of the argument that led to the GPD in the univariate case (Ferreira and de Haan, 2014)
- Let $\{\tilde{Y}(s)\}_{s \in S}$ be a stochastic process with Pareto(1). The limit

$$\left\{ \frac{\tilde{Y}(s)}{u} \middle| \sup_{s \in S} \tilde{Y}(s) > u \, , \, s \in S \right\} \xrightarrow{D} \{Z(s), \, s \in S\}, \qquad u \to \infty,$$

$\{Z(s)\}$ is called a **standard Pareto process** and

$$\{Z(s), s \in S\} \stackrel{\mathrm{D}}{=} \{RW(s), s \in S\} \, .$$

where $R \sim$ Pareto(1) is independent of $W(s) \geq 0$, and $\sup_{s \in S} W(s) \stackrel{\text{a.s.}}{=} 1$

New model ?

- Dependence structure as in the max-stable processes
- ... but more available data for inference and a simplified representation, based on a single representation

$$Z(s) = RW(s)$$

- Threshold stability property

$$\Pr(Z(s)/u \in B | \sup_{s \in S} Z(s) > u) = \Pr(Z(s) \in B)$$

$\Rightarrow$ we may suppose that for large enough thresholds u on the Pareto scale,

$$\left. \frac{\tilde{Y}(s)}{u} \right| \sup_{s \in S} \tilde{Y}(s) > u \overset{\mathrm{D}}{\approx} Z(s)$$

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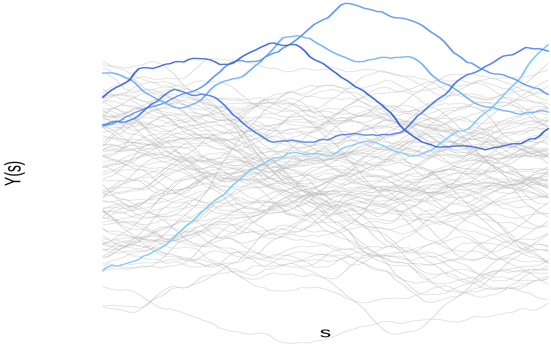
Looking at other conditioning sets... I

- Conditioning event $\{\sup_{s\in S} Y(s) > u\}$, for large u

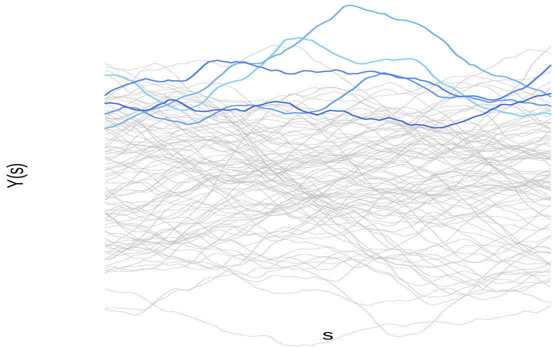
$$\{Y(s), s \in S \mid \sup_{s\in S} Y(s) > u\}$$



Looking at other conditioning sets... II

- Conditioning event $\{\inf_{s \in S} Y(s) > u\}$, for large u

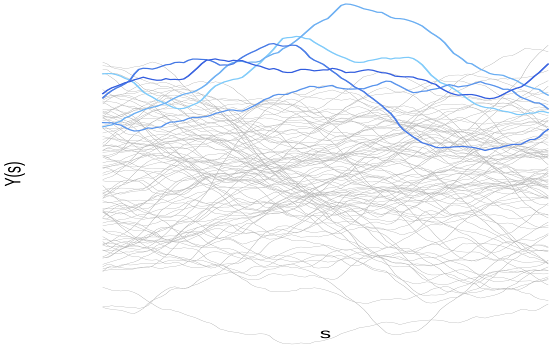
$$\{Y(s), s \in S \mid \inf_{s \in S} Y(s) > u\}$$



Looking at other conditioning sets... III

- Conditioning event $\{\int_{s\in S} Y(s)\, ds > u\}$, for large u

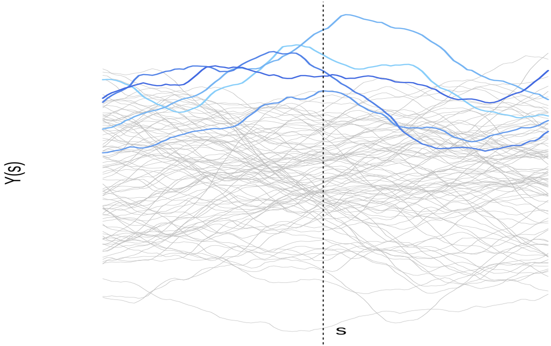
$$\{Y(s),\, s \in S \mid \int_{s\in S} Y(s)\, ds > u\}$$



Looking at other conditioning sets... IV

- Conditioning event $\{Y(s_0) > u\}$, for large u

$$\{Y(s),\ s \in S \mid Y(s_0) > u\}$$



ℓ-Pareto and r-Pareto processes

- Extension of GP processes (Dombry and Ribatet, 2015; de Fondeville and Davison, 2022)
- Functional like as $\sup_{s \in S} Y(s)$, $\int_{s \in S} Y(s) ds$,...
- r (or ℓ) risk homogeneous functional

$$r(cY) = cr(Y), \qquad c > 0$$

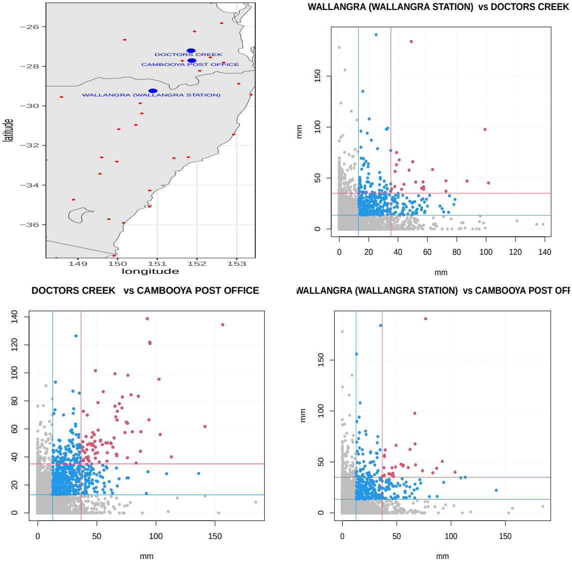
- Transform the data to ensure a common Pareto distribution, and work on exceedances defined on this transformed scale $\{\tilde{Y}(s)\}$ ... **but many extreme phenomena are most naturally characterized on the scale of the original data.**

$$\left. \frac{\tilde{Y}(s)}{u} \right| r(\tilde{Y}) > u \overset{\mathrm{D}}{\approx} RW(s), \quad \text{with } r(W) \overset{\text{a.s.}}{=} 1$$

Subasymptotic models for spatial extremes

Huser, R. and Wadsworth, J. (2022) Advances in statistical modeling of spatial extremes, *WIREs Computational Statistics*, 14:e1537.

Australian rainfall data: joint upper tail



Pairwise extremal dependence

- Assume that $Y(s)$ with common marginal distribution function $F(y) = \Pr(Y(s) \leq y)$ which has upper endpoint y_F.
- Summary measure for extremal dependence of pair observations $Y(s)$, $Y(s')$ when $u \simeq y_F$.

$$\chi_u(s, s') = \Pr(Y(s) > u | Y(s') > u) = \frac{\Pr(Y(s) > u, Y(s') > u)}{\Pr(Y(s') > u)}$$

- If $Y(s)$ and $Y(s')$ are independent

$$\chi_u(s, s') = \Pr(Y(s) > u | Y(s') > u) = \Pr(Y(s) > u)$$

Coefficient of upper tail: χ I

- **Tail dependence**. Consider the limit

$$\chi_u(s, s') \to \chi(s, s'), \qquad \text{as } u \to \infty.$$

- $0 \leq \chi(s, s') \leq 1$
- If for all $s \neq s'$, $\chi(s, s') > 0$ the process is asymptotically dependent.
- If for all $s \neq s'$, $\chi(s, s') = 0$ the process is asymptotic independence.

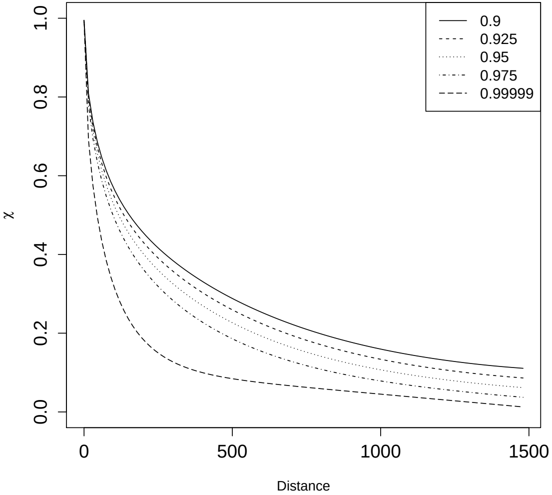
Coefficient of upper tail: χ II

Examples:

- Max-stable stationary processes: asymptotically dependent for $\|s - s'\| \leq h^*$, independent for $\|s - s'\| > h^*$
- Gaussian stationary processes are asymptotically independent

Empirical extremal dependence measure: $\hat{\chi}_u(s, s')$

We inspect several thresholds u and distances $\|s - s'\|$



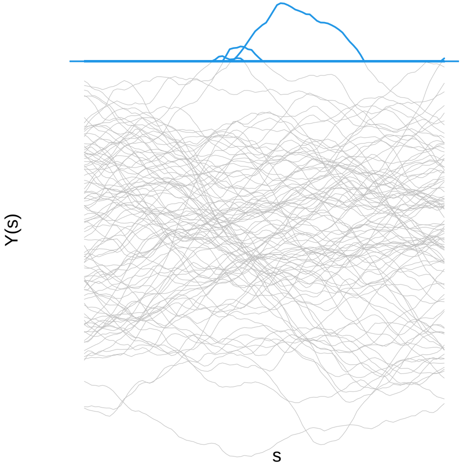
Smoothed values of the empirical estimates

Features of the data

- Dependence weakens as
  - distance increases
  - threshold increases
- Asymptotic dependence or independence?
- Different dependence structures at different lags

Models for exceedances of $Y(s)$

- Models for $\{\max(Y(s) - u, 0),\ s \in S\}$ with u large



- We look at models that allow for both dependence classes

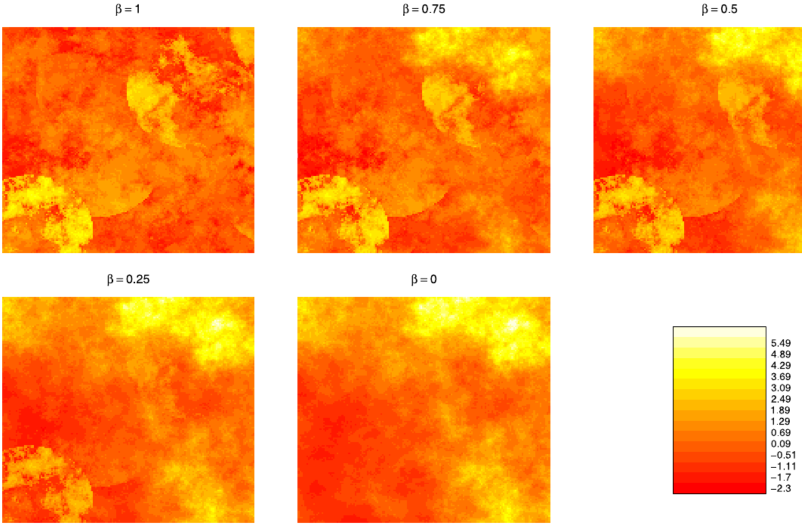
Hybrid models

- Wadsworth and Tawn (2012), Bacro et al. (2016)

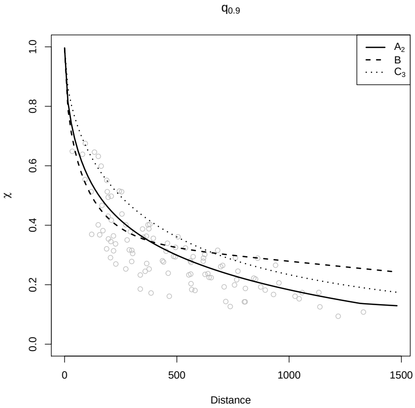
$$Y(s) = \max\left(\beta AD(s), (1-\beta)AI(s)\right), \quad \beta \in [0,1]$$

- $AD(s)$ an asymptotically dependent process up to a fixed distance
- $AI(s)$ an asymptotically independent process
- Independent and both with unit Fréchet margins.

Hybrid models: simulation



Australian data rainfall



A_2: **Hybrid**, B: **AD**, C_3: **AI**

Model for uncertain extremal dependence

- Huser and Wadsworth (2019)

$$Y(s) = R^{\delta} \left\{ AI(s) \right\}^{1-\delta}, \delta \in [0,1]$$

- $R \sim$ Pareto(1)
- $AI(s) \sim$ Pareto(1) marginally

$$\chi_u(s,s') \to \chi(s,s') = \begin{cases} > \mathbf{0} & \text{for } \delta > 1/2 \\ = \mathbf{0} & \text{for } \delta > 1/2 \end{cases}, \qquad \text{as } u \to \infty.$$

No more time for ...

More models

- Max-infinitely divisible processes (Huser et al., 2021)
- Conditional spatial extremes model (Wadsworth and Tawn, 2022)

Estimation

- Max-stable (Davison et al., 2012)
- Exceedances (de Fondeville and Davison, 2022)

Simulation

- Unconditional simulation (Oesting et al., 2012)
- Conditional simulation (Dombry et al., 2013)

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